

A New Minimisation Principle for Poisson Equation Leading to a Flexible Finite Element Approach

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Abstract

We introduce a new minimisation principle for Poisson equation using two variables: the solution and the gradient of the solution. This principle allows us to use any conforming finite element spaces for both variables, where the finite element spaces do not need to satisfy a so-called inf-sup condition. A numerical example demonstrates the superiority of the approach.

Key words: Poisson equation, minimisation principle, mixed finite element method, a priori error estimate

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1 Introduction

It is often more important to get the accurate approximation of the gradient of the solution of a Poisson equation. In that case, a mixed formulation of the Poisson equation is used, where there are two unknowns - the solution and its gradient - in the variational equation. Discretising a mixed formulation of a partial differential equation is a challenging task as the involved finite element spaces should satisfy a compatibility condition - so called inf-sup condition. Although there are many finite element spaces discovered satisfying the compatibility condition for the Poisson equation [2, 6–9, 13, 16], it is not so easy for mixed formulations of other partial differential equations. It is sometimes

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useful to use a least-squares finite element method to approximate the solution and its gradient simultaneously [3–5]. A least-squares formulation allows the use of any conforming finite element spaces avoiding the compatibility condition.

In this paper, we propose a new minimisation principle for Poisson equation using the solution and the gradient of the solution as two unknowns. This formulation is similar to a least-squares formulation in the sense that it allows the use of any conforming finite element spaces avoiding the compatibility condition [3–5, 11]. However, in comparison to a least-squares finite element method, the source term f can be in the dual of H^1 -space, and the gradient can be discretised using a L^2 -conforming finite element space. We also prove optimal a priori error estimates for the proposed finite element method.

The structure of the rest of the paper is organised as follows. In the next section, we introduce our formulation and show its well-posedness. We propose finite element methods for the given formulation and prove a priori error estimates in Section 3. A numerical example with discretisation errors are presented in Section 4 and a short conclusion is drawn in Section 5.

2 A New Formulation of Poisson equation

In this section we introduce a new minimisation principle of the Poisson problem. Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be a bounded convex domain with polygonal or polyhedral boundary $\partial\Omega$ with the outward pointing normal \mathbf{n} on $\partial\Omega$.

We start with the following minimisation problem for the Poisson problem

Problem 1. *Given $f \in H^{-1}(\Omega)$ we want to find*

$$u = \arg \min_{v \in H_0^1(\Omega)} K(v) \quad (1)$$

with

$$K(v) = \frac{1}{2} \int_{\Omega} |\nabla v|^2 dx - \ell(v). \quad (2)$$

where

$$\ell(v) = \int_{\Omega} f v dx.$$

We refer to the following references [1, 2, 6–9, 12, 13, 15, 16] for different variational formulations of Poisson equation.

Let $V = H_0^1(\Omega)$ and $\mathbf{Q} = [L^2(\Omega)]^d$, and for two vector-valued functions $\boldsymbol{\alpha} : \Omega \rightarrow \mathbb{R}^d$ and $\boldsymbol{\beta} : \Omega \rightarrow \mathbb{R}^d$, the Sobolev inner product on the Sobolev space $H^k(\Omega)$ ($k \in \mathbb{R}$) be defined as

$$\langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle_{k,\Omega} := \sum_{i=1}^d \langle \alpha_i, \beta_i \rangle_{k,\Omega},$$

where $(\boldsymbol{\alpha})_i = \alpha_i$, $(\boldsymbol{\beta})_i = \beta_i$ with $\alpha_i, \beta_i \in H^k(\Omega)$, for $i = 1, \dots, d$, and the norm $\|\cdot\|_{H^k(\Omega)}$ is induced from this inner product. We will use the standard notation $\|\cdot\|_{k,\Omega}$ for the norm in the $H^k(\Omega)$ -space. We now introduce a functional $J_{\alpha,\gamma}(v, \boldsymbol{\tau}; f)$ with

$$J_{\alpha,\gamma}(v, \boldsymbol{\tau}; f) = \|\boldsymbol{\tau}\|_{0,\Omega}^2 + \|\boldsymbol{\tau} - \alpha \nabla v\|_{0,\Omega}^2 + \gamma \ell(v),$$

where $\alpha > 0$ and γ are two fixed constants, and consider another minimisation problem for two variables $(v, \boldsymbol{\tau}) \in [V \times \mathbf{Q}]$

$$\arg \min_{(v, \boldsymbol{\tau}) \in [V \times \mathbf{Q}]} J_{\alpha,\gamma}(v, \boldsymbol{\tau}; f). \quad (3)$$

The minimisation problem is equivalent to finding $(u, \boldsymbol{\sigma}) \in [V \times \mathbf{Q}]$ such that

$$a((u, \boldsymbol{\sigma}), (v, \boldsymbol{\tau})) = -\frac{\gamma}{2} \ell(v), \quad (v, \boldsymbol{\tau}) \in [V \times \mathbf{Q}], \quad (4)$$

where the bilinear form $a(\cdot, \cdot)$ is defined as

$$a((u, \boldsymbol{\sigma}), (v, \boldsymbol{\tau})) = (\boldsymbol{\sigma}, \boldsymbol{\tau})_{0,\Omega} + (\boldsymbol{\sigma} - \alpha \nabla u, \boldsymbol{\tau} - \alpha \nabla v)_{0,\Omega}.$$

Standard arguments can be used to show the continuity of the bilinear form $a(\cdot, \cdot)$ on the space $V \times \mathbf{Q}$. Now we show that the bilinear form $a(\cdot, \cdot)$ is coercive on $V \times \mathbf{Q}$.

Lemma 1. *Let α and γ be two constants with $\alpha > 0$. For $(u, \boldsymbol{\sigma}) \in [V \times \mathbf{Q}]$ the bilinear form $a(\cdot, \cdot)$ satisfies*

$$a((u, \boldsymbol{\sigma}), (u, \boldsymbol{\sigma})) \geq \frac{\alpha}{\alpha + 2C_1} (\|u\|_{1,\Omega}^2 + \|\boldsymbol{\sigma}\|_{0,\Omega}^2),$$

where C_1 is the constant in the Poincaré inequality

$$\|u\|_{1,\Omega}^2 \leq C_1 \|\nabla u\|_{0,\Omega}^2.$$

Proof. The proof follows from a triangle inequality and Poincaré inequality:

$$\begin{aligned}
\|u\|_{1,\Omega}^2 + \|\boldsymbol{\sigma}\|_{0,\Omega}^2 &\leq \frac{C_1}{\alpha} \|\alpha \nabla u\|_{0,\Omega}^2 + \|\boldsymbol{\sigma}\|_{0,\Omega}^2 \\
&\leq \frac{2C_1}{\alpha} [\|\boldsymbol{\sigma} - \alpha \nabla u\|_{0,\Omega}^2 + \|\boldsymbol{\sigma}\|_{0,\Omega}^2] + \|\boldsymbol{\sigma}\|_{0,\Omega}^2 \\
&\leq \frac{2C_1 + \alpha}{\alpha} (\|\boldsymbol{\sigma}\|_{0,\Omega}^2 + \|\boldsymbol{\sigma} - \alpha \nabla u\|_{0,\Omega}^2) \\
&= \frac{2C_1 + \alpha}{\alpha} a((u, \boldsymbol{\sigma}), (u, \boldsymbol{\sigma})).
\end{aligned}$$

□

Corollary 1. *Since the bilinear form $a(\cdot, \cdot)$ is continuous and coercive on $V \times \mathbf{Q}$, and the linear form $\ell(v)$ is also continuous on V for $f \in H^{-1}(\Omega)$, the problem of finding $(u, \boldsymbol{\sigma}) \in [V \times \mathbf{Q}]$ such that*

$$a((u, \boldsymbol{\sigma}), (v, \boldsymbol{\tau})) = -\frac{\gamma}{2} \ell(v), \quad (v, \boldsymbol{\tau}) \in [V \times \mathbf{Q}], \quad (5)$$

has a unique solution from Lax-Milgram lemma.

Remark 1. *In contrast to the standard least-squares method, where we need to have $\ell \in L^2(\Omega)$, we have here $f \in H^{-1}(\Omega)$. Thus the standard least-squares method cannot handle the situation if the source function is not L^2 , whereas the new approach requires exactly the same regularity for f as the standard Galerkin approach.*

Let $(u_e, \boldsymbol{\sigma}_e) \in V \times \mathbf{Q}$ be the solution of the minimisation problem (3). We now choose α and γ in such a way that the solution $(u, \boldsymbol{\sigma})$ of the minimisation problem (3) satisfies $u = u_e$ and $\boldsymbol{\sigma}_e = \nabla u$. Here the natural norm for an element $(v, \boldsymbol{\tau}) \in V \times \mathbf{Q}$ of the product space $V \times \mathbf{Q}$ is $\sqrt{\|v\|_{1,\Omega}^2 + \|\boldsymbol{\tau}\|_{0,\Omega}^2}$. Thus (3) leads to the problem of finding $(u, \boldsymbol{\sigma}) \in [V \times \mathbf{Q}]$ such that

$$2(\boldsymbol{\sigma}, \boldsymbol{\tau})_{0,\Omega} + 2(\boldsymbol{\sigma} - \alpha \nabla u, \boldsymbol{\tau} - \alpha \nabla v)_{0,\Omega} + \gamma \ell(v) = 0, \quad (v, \boldsymbol{\tau}) \in [V \times \mathbf{Q}].$$

Letting the test functions $\boldsymbol{\tau} = \mathbf{0}$ and $v = 0$ successively in the above equation leads to

$$\begin{aligned}
-2(\boldsymbol{\sigma} - \alpha \nabla u, \alpha \nabla v)_{0,\Omega} + \gamma \ell(v) &= 0, \quad v \in V, \\
(\boldsymbol{\sigma}, \boldsymbol{\tau})_{0,\Omega} + (\boldsymbol{\sigma} - \alpha \nabla u, \boldsymbol{\tau})_{0,\Omega} &= 0, \quad \boldsymbol{\tau} \in \mathbf{Q}.
\end{aligned} \quad (6)$$

The second equation immediately yields

$$(2\boldsymbol{\sigma} - \alpha \nabla u, \boldsymbol{\tau})_{0,\Omega} = 0, \quad \boldsymbol{\tau} \in \mathbf{Q},$$

and hence $\alpha = 2$ ensures that $\boldsymbol{\sigma} = \nabla u$. Using $\boldsymbol{\sigma} = \nabla u$ in the first equation of (6), we have

$$-2\alpha(1 - \alpha)(\nabla u, \nabla v)_{0,\Omega} + \gamma\ell(v) = 0.$$

We have the standard variational problem for the Poisson equation if $\gamma = 2\alpha(1 - \alpha)$, and thus setting $\alpha = 2$, we get $\gamma = -4$. Now we have the following problem.

Problem 2. *Given $f \in H^{-1}(\Omega)$, the variational equation for the minimisation problem is to find $(u, \boldsymbol{\sigma}) \in [V \times \mathbf{Q}]$ such that*

$$a((u, \boldsymbol{\sigma}), (v, \boldsymbol{\tau})) = 2\ell(v), \quad (v, \boldsymbol{\tau}) \in [V \times \mathbf{Q}], \quad (7)$$

where the bilinear form $a(\cdot, \cdot)$ is defined as

$$a((u, \boldsymbol{\sigma}), (v, \boldsymbol{\tau})) = (\boldsymbol{\sigma}, \boldsymbol{\tau})_{0,\Omega} + (\boldsymbol{\sigma} - 2\nabla u, \boldsymbol{\tau} - 2\nabla v)_{0,\Omega}.$$

From the above discussion we have the following proposition.

Proposition 1. *Let u be the solution of Problem 1 and $(\tilde{u}, \tilde{\boldsymbol{\sigma}})$ of Problem 2. Then we have $\tilde{u} = u$ and $\tilde{\boldsymbol{\sigma}} = \nabla u$.*

Remark 2. *The idea can be easily generalised to a general differential equation, which can be put in a minimisation framework. For example, consider the solution of the linear elastic problem of finding the displacement field $\mathbf{u} \in \mathbf{V} = [H_0^1(\Omega)]^d$ such that [6]*

$$\mathbf{u} = \arg \min_{\mathbf{v} \in \mathbf{V}} \frac{1}{2} \int_{\Omega} \varepsilon(\mathbf{v}) : \mathcal{C} \varepsilon(\mathbf{v}) dx - \ell(\mathbf{v}),$$

where $\varepsilon(\mathbf{v}) = \frac{1}{2} (\nabla \mathbf{v} + [\nabla \mathbf{v}]^T)$ is the symmetric part of the gradient, \mathcal{C} is the Hooke's tensor, and $\ell(\cdot)$ is a linear form

$$\ell(\mathbf{v}) = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} dx.$$

By defining a pseudo-stress $\boldsymbol{\sigma} = \sqrt{\mathcal{C}} \varepsilon(\mathbf{v})$, we can put this in the above framework with

$$a(\mathbf{u}, \boldsymbol{\sigma}, \mathbf{v}, \boldsymbol{\tau}) = (\boldsymbol{\sigma}, \boldsymbol{\tau})_{0,\Omega} + (\boldsymbol{\sigma} - 2\sqrt{\mathcal{C}} \varepsilon(\mathbf{u}), \boldsymbol{\tau} - 2\sqrt{\mathcal{C}} \varepsilon(\mathbf{v}))_{0,\Omega}.$$

We note that since \mathcal{C} is a symmetric positive definite tensor, its square is well-defined.

3 Finite element approximation and a priori error estimate

Let \mathcal{T}_h be a quasi-uniform partition of the domain Ω in simplices, convex quadrilaterals or hexahedra having the mesh-size h . Let \hat{T} be a reference simplex or square or cube, where the reference simplex is defined as

$$\hat{T} := \{\mathbf{x} \in \mathbb{R}^d : x_i > 0, i = 1, \dots, d, \text{ and } \sum_{i=1}^d x_i < 1\},$$

and the reference square or cube $\hat{T} := (0, 1)^d$.

The finite element space is defined by affine maps F_T from a reference element \hat{T} to a physical element $T \in \mathcal{T}_h$. For $k \geq 0$, let $\mathcal{Q}_k(\hat{T})$ be the space of polynomials of degree less than or equal to k in \hat{T} in the variables x_1, \dots, x_d if \hat{T} is the reference simplex, the space of polynomials in \hat{T} of degree less than or equal to k with respect to each variable x_1, \dots, x_d if \hat{T} is the reference square or cube.

Then the finite element space based on the mesh \mathcal{T}_h is defined as the space of continuous functions whose restrictions to an element T are obtained by maps of given polynomial functions from the reference element; that is,

$$S_h := \left\{ v_h \in H^1(\Omega) : v_h|_T = \hat{v}_h \circ F_T^{-1}, \hat{v}_h \in \mathcal{Q}_k(\hat{T}), T \in \mathcal{T}_h \right\},$$

see [6, 7, 10, 14]. We now define $V_h := S_h \cap H_0^1(\Omega)$. We also want to define two other finite element spaces L_h and \mathbf{Q}_h as

$$L_h := \{v_h \in L^2(\Omega) : v_h|_T \in \mathcal{Q}_k(T), T \in \mathcal{T}_h\}, \quad \mathbf{Q}_h := [L_h]^d,$$

Now a discrete formulation of our problem is to find $(u_h, \boldsymbol{\sigma}_h) \in [V_h \times \mathbf{Q}_h]$ such that

$$a((u_h, \boldsymbol{\sigma}_h), (v_h, \boldsymbol{\tau}_h)) = 2\ell(v_h), \quad (v_h, \boldsymbol{\tau}_h) \in [V_h \times \mathbf{Q}_h]. \quad (8)$$

Since $[V_h]^d \subset \mathbf{Q}$, we can also use $\mathbf{V}_h = [V_h]^d$ to discretize the gradient of the continuous problem. This leads to a problem of finding $(u_h, \boldsymbol{\sigma}_h) \in [V_h \times \mathbf{V}_h]$ such that

$$a((u_h, \boldsymbol{\sigma}_h), (v_h, \boldsymbol{\tau}_h)) = 2\ell(v_h), \quad (v_h, \boldsymbol{\tau}_h) \in [V_h \times \mathbf{V}_h], \quad (9)$$

which utilizes equal order interpolation. Since the discrete formulation is conforming, the bilinear form $a(\cdot, \cdot)$ and the linear form $\ell(\cdot)$ are both continuous on the corresponding spaces. The coercivity also follows from the continuous setting.

Theorem 1. *Thus the discrete problem of finding $(u_h, \sigma_h) \in [V_h \times \mathbf{Q}_h]$ or $(u_h, \sigma_h) \in [V_h \times \mathbf{V}_h]$ such that*

$$a((u_h, \sigma_h), (v_h, \tau_h)) = 2\ell(v_h), \quad (v_h, \tau_h) \in [V_h \times \mathbf{Q}_h] \quad \text{or} \quad (v_h, \tau_h) \in [V_h \times \mathbf{V}_h]$$

has a unique solution, and the solution satisfies

$$\|u - u_h\|_{1,\Omega} + \|\sigma - \sigma_h\|_{0,\Omega} \leq c \left(\inf_{v_h \in V_h} \|u - v_h\|_{1,\Omega} + \inf_{\tau_h \in \mathbf{V}_h \text{ or } \tau_h \in \mathbf{Q}_h} \|\sigma - \tau_h\|_{0,\Omega} \right),$$

where u is the exact solution of the problem (1) and $\sigma = \nabla u$.

Proof. The proof follows from Galerkin orthogonality and standard arguments. \square

Remark 3. *The solution u is assumed to be in $H_0^1(\Omega)$ only for the purpose of simplicity. In fact, any non-zero Dirichlet condition or mixture of Dirichlet and Neumann boundary conditions are all fine as in the case of the standard Galerkin finite element method.*

4 Numerical example

In this section we consider a numerical example to demonstrate the performance of this new minimisation scheme. In fact, we show the discretisation errors for the solution u in the L^2 and H^1 -norms, and discretisation errors for the gradient in the L^2 -norm. For this example, we consider the domain of the square $\Omega = [-1, 1]^2$ with the exact solution

$$u(x, y) = ((x - y) \exp(-5.0(x - 0.5)(x - 0.5) - 5.0(y - 0.5)(y - 0.5))),$$

where the right-hand side function f and the Dirichlet boundary condition on $\partial\Omega$ is obtained by using this exact solution. The two components of the gradient are denoted by σ^1 and σ^2 , respectively, where their numerical approximations are denoted by σ_h^1 and σ_h^2 , respectively. We start with the initial uniform triangulation of 32 triangles in the first level and then refine uniformly in each level. We have tabulated the discretisation errors in Table 1. We can see that the numerical results are as predicted by the theory. Moreover, the discretisation errors show the superiority of the scheme as the discretisation errors for the gradient of the solution converge quadratically to the exact solution. This is normally not achieved in any mixed finite element methods.

Table 1: Discretisation errors for the solution and gradient

l	$\ u - u_h\ _{1,\Omega}$		$\ u - u_h\ _{0,\Omega}$		$\ \sigma^1 - \sigma_h^1\ _{0,\Omega}$		$\ \sigma^2 - \sigma_h^2\ _{0,\Omega}$	
1	3.41208e-01		3.71839e-02		1.72948e-01		1.72948e-01	
2	1.70261e-01	1.00	1.15857e-02	1.68	5.39510e-02	1.68	5.39510e-02	1.68
3	8.40661e-02	1.02	3.09011e-03	1.91	1.45061e-02	1.89	1.45061e-02	1.89
4	4.18485e-02	1.01	7.86293e-04	1.97	3.74056e-03	1.96	3.74056e-03	1.96
5	2.08998e-02	1.00	1.97503e-04	1.99	9.62306e-04	1.96	9.62306e-04	1.96
6	1.04469e-02	1.00	4.94393e-05	2.00	2.51753e-04	1.93	2.51753e-04	1.93

5 Conclusion

We have proposed a new minimisation principle for the Poisson equation based on the solution and its gradient. One big advantage of this formulation is that a finite element approximation can be performed as in a least-squares finite element method without fulfilling the compatibility condition between two finite element spaces. However, the finite element approach is much easier than in a least-squares approach. An optimal a priori error estimate is proved for the proposed formulation. A numerical example is presented to demonstrate the optimality of the scheme.

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